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On the equivalence of certain statements in triangles

Sobre la equivalencia de ciertas afirmaciones en los triángulos

Sobre a equivalência de certas afirmações em triângulos

[1]Abasov Raghib Zeinal[*]

[1]Azerbaijan State Pedagogical University. ORCID: https://orcid.org/0009-0009-8217-9916

[*]Autor para la correspondencia: abasov50@inbox.ru

Abstract

The equivalence of statements is one of the important concepts in mathematics. This concept plays an important role in solving some problems in both higher and school mathematics. However, sometimes these concepts are applied to problems that have different structures. In this article, general aspects are discussed to demonstrate the equivalence of some statements, as well as to solve some other problems on the topic of triangles. Furthermore, potential difficulties in this process are highlighted that still remain open problems in this field of research.

Keywords: triangles; equivalent statements; transformations; equivalence of systems

Resumen

La equivalencia de enunciados es uno de los conceptos más importantes de las matemáticas. Este concepto juega un papel importante en la resolución de algunos problemas tanto en matemáticas superiores como en matemáticas escolares. Sin embargo, a veces estos conceptos se aplican a problemas que tienen estructuras diferentes. En este artículo se discuten aspectos generales para demostrar la equivalencia de algunos enunciados, así como para resolver algunos otros problemas sobre el tema de los triángulos. Además, se destacan potenciales dificultades en este proceso que todavía permanecen como problemas abiertos en este campo de investigación.

Palabras clave: triángulos; enunciados equivalentes; transformaciones; equivalencia de sistemas

Resumo

A equivalência de enunciados é um dos conceitos importantes da matemática. Este conceito desempenha um papel importante na resolução de alguns problemas tanto de matemática superior quanto de matemática escolar. No entanto, às vezes esses conceitos são aplicados a problemas que possuem estruturas diferentes. Neste artigo são discutidos aspectos gerais para demonstrar a equivalência de algumas afirmações, bem como para resolver alguns outros problemas sobre o tema triângulos. Além disso, são destacadas potenciais dificuldades neste processo que ainda permanecem problemas em aberto neste campo de pesquisa.

Palavras-chave: triângulos; enunciados equivalentes; transformações; equivalência de sistemas

Introduction

Geometry is the branch of mathematics concerned with the shape of individual objects, spatial relationships among various objects, and the properties of surrounding space. It is one of the oldest branches of mathematics, having arisen in response to such practical problems as those found in surveying. In fact, its etymological roots are derived from Greek words meaning "Earth measurement". However, over time, the scope of geometry extended beyond the investigation of flat surfaces (commonly referred to as plane geometry) and rigid three-dimensional structures (known as solid geometry). Instead, it was acknowledged that even the most abstract concepts and mental representations could be expressed and developed through geometric principles (Heilbron, 2023). Thus, today there are various branches of geometry including differential geometry, algebraic

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geometry, and geometric analysis which are characterized by the objects they study and the methods that are used in this purpose.

As pointed out by Dillon (2018) many mathematicians who specialize in geometry do not talk about similar triangles or alternate interior angles or angles in a circle or most of the other things people think about school geometry, that is, Euclidean geometry. But as in many other fields of science, knowledge is built on the basis of previous discoveries, from the simple to the complex, and although sometimes dismissed by students, the study of geometry and its demonstrations has a profound impact. For example, the utilization of demonstrations can serve as a means to elucidate the points of convergence between mathematical knowledge and personal experience within the practice of "engaging in mathematics." This approach facilitates the amalgamation of one's cognitive awareness, rendering the underlying purpose more transparent, dissecting the entirety into constituent components, transforming the implicit into the explicit, and advancing incrementally through the various stages of problem-solving (Barra et al. 2009; Ingram, 2021).

Within geometry, a triangle represents a fundamental, characterized as a polygon possessing three sides, thereby constituting a closed, two-dimensional figure composed of straight segments. Within the domain of geometry, triangles hold a distinctive status owing to several inherent properties: 1) the sum of the internal angles of any triangle invariably totals 180 degrees, 2) in a triangle, the length of each side is invariably less than the sum of the remaining two sides, a principle known as the triangle inequality. Triangles can be categorized based on two primary criteria: the lengths of their sides and the magnitudes of their angles. This classification gives rise to several distinct types of triangles, including equilateral, isosceles, scalene, acute, right, and obtuse triangles. The application of triangles extends across diverse fields such as mathematics, engineering, and architecture, where they find relevance in addressing a wide array of real-world problems and calculations. In fact, Clark and Pathania (2023) include triangle congruence as one of the foundational principles of Euclidean geometry.

Considering the above, the objective of this work is to discuss some open questions in the equivalence of certain statements in triangles. We would like to point out that the cases analyzed show considerable complexity, however these types of open problems are found more frequently in practical activities, so a school mathematics course should help students develop the ability to solve them.

Materials and methods

The main method used is direct demonstration. The method of direct demonstration, grounded in axiomatic principles, constitutes a means of establishing the validity or invalidity of mathematical propositions through the logical integration of established facts, typically rooted in axioms. Axioms are foundational truths within a given mathematical system that necessitate no further proof, serving as the bedrock upon which all subsequent theorems are constructed. Then, a direct proof represents a systematic and logical progression of statements designed to demonstrate the veracity or falsity of a given assertion. This method harnesses a concatenation of simple statements, commencing with the hypothesis and culminating in the desired conclusion, while making use of axioms, definitions, postulates, theorems, and lemmas.

Direct proof is recognized as the most elementary form of mathematical proof and is meticulously assembled by orchestrating a series of logical steps, wherein the conclusion is ultimately underpinned by the harmonious synthesis of axioms, definitions, and previously established theorems. Consequently, direct demonstration, founded upon axiomatic principles, stands as a pivotal and fundamental technique for substantiating mathematical statements.

Results and discussion

Proving the equivalence of certain statements has always been highly appreciated not only in school mathematics, but throughout the scientific world. These proofs are considered more valuable when the proven (equivalent) statements have different structures. It is clear that the equivalence of different statements, generally speaking, is proved in different ways. And if several different statements are proved in the same way, this is considered very effective. In this work, the equivalence of some different statements from the topic of triangles is proved in the same way.

Recently, I accidentally saw one small note by (Chernikov, 2003, p.43) where the equivalence of the following two problems is proved:

<u>Problem 1.</u> Let us denote by a, b, ca, b, c the lengths of the sides of a certain triangle. Let's prove that

$a^2(b+ca)+b^2(c+ab)+c^2(a+bc)<3abc \quad (1)$

$a^2(b+ca)+b^2(c+ab)+c^2(a+bc)<3abc \quad (1)$.

<u>Problem 2.</u> Prove that for angles A, B, CA, B, C of any triangle the following inequality is true: $cos(A)+cos(B)+cos(C)\leq\frac{3}{2} \quad (2) cos(A)+cos(B)+cos(C)\leq\frac{3}{2} \quad (2)$. (Morozova & Petrakova, 1971).

As can be seen, these tasks have different structures. To prove these problems, the author uses the following method. Let's assume that problem 2 has been solved. We must prove (1) using (2). Notice, that:

$$a^2bc + b^2ac + c^2ab = a^3 + b^3 + c^3 - 3abc = (a+b+c) \times (a^2 + b^2 + c^2 - ab - bc - ac) = a(b^2 + c^2 - a^2) + b(a^2 + c^2 - b^2) + c(a^2 + b^2 - c^2) \quad (3)$$

Then inequality (1) is reduced to the form:

$$a(b^2 + c^2 - a^2) + b(a^2 + c^2 - b^2) + c(a^2 + b^2 - c^2) < 3abc \quad (4)$$

So, we need to prove that:

$$\frac{(b^2 + c^2 - a^2)}{2bc} + \frac{(a^2 + c^2 - b^2)}{2ac} + \frac{(a^2 + b^2 - c^2)}{2ab} < \frac{3}{2} \quad (5)$$

Using the cosine theorem from (5) we obtain (2). On the contrary, let us assume that Problem 1 has been solved. By the cosine theorem, we have:

$$cos(A) = \frac{(b^2 + c^2 - a^2)}{2bc} \quad cos(B) = \frac{(a^2 + c^2 - b^2)}{2ac} \quad cos(C) = \frac{(a^2 + b^2 - c^2)}{2ab} \quad (6)$$

Then, considering the equalities in (4) we got (2). These problems and their solutions require some creativity. But, nevertheless, guessing equality (3) is almost random by nature and it is checked by direct calculation by opening the brackets. Notice that the solution is based precisely on inequality (3). Then, what happens if we fail to find equality (3)? In this regard is important to point out that there are quite a few such problems among the Olympiad problems. What to do then? How to find a universal, constructive method for solving a given problem? This topic in geometry is quite large, rich, and full of variety, as well as Olympiad problems.

In the book by Prasolov (1986) the author talked about just one such technique, the initial basis of which is the problem 15.9 from Lyapin et al. 1973.

<u>Problem 3.</u> Let a, b, ca, b, c be given numbers. Prove that the existence of positive numbers x, y, z x, y, z for which the equalities hold:

$$\begin{cases} a = y + z \\ b = x + z \\ c = x + y \end{cases} \quad (7)$$

Is a necessary and sufficient condition that the numbers a, b, ca, b, c be the lengths of the sides of some $\Delta ABC \Delta ABC$. Lyapin et al. 1973, proposed the following solution from system (1). By simple calculation we find that:

$$\begin{cases} x = \dfrac{(b+c-a)}{2} \\ y = \dfrac{(a+c-b)}{2} \\ z = \dfrac{(a+b-c)}{2} \end{cases} \quad (8)$$

System (7) is also easily obtained from (8), i.e., systems (7) and (8) are equivalent. Indeed, let a, b, c a, b, c be the lengths of the sides $\Delta ABC \Delta ABC$. Then, using the triangle inequality, we can state that from (8) the numbers x, y, zx, y, z found will be positive. On the contrary, if x, y, zx, y, z are positive numbers, then from system (7) the found numbers a, b, ca, b, c will be positive and $a + b = x + y + 2z \Rightarrow a + b > ca + b = x + y + 2z \Rightarrow a + b > c$ and also $a - b = yx \Rightarrow |a - b| = \lfloor y - x \rfloor < y + x = c \Rightarrow |a - b| < c$ $a - b = yx \Rightarrow |a - b| = \lfloor y - x \rfloor < y + x = c \Rightarrow |a - b| < c$ etc. That is, the numbers a, b, ca, b, c satisfy the triangle inequality and they can be the lengths of the sides of some triangle.

Thus, knowing the values a, b, ca, b, c from (8), we find new values x, y, zx, y, z. How do these two systems of numbers differ and what benefit do new values of x, y, zx, y, z bring when solving (or proving) similar problems as Problem 1 and Problem 2 and for what reason does this effect occur? Our answer is:

1) a, b, ca, b, c are the lengths of the sides of a certain triangle, which means they are positive and satisfy the triangle inequalities. This means that when we solve some problem on the topic of triangles, and we must take this fact into account in all reasoning in solving this problem. In this sense, this fact plays the role of an additional load on a, b, ca, b, c because these quantities are not free.

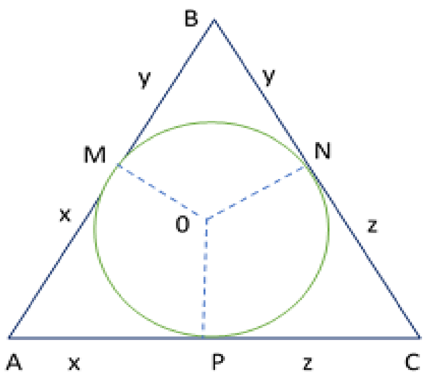
2) On the contrary, the quantities x, y, zx, y, z are also positive, but unlike a, b, ca, b, c, they do not have to satisfy the triangle inequalities (this is the main difference from a, b, ca, b, c). Let's check this in a specific example. Let's build a triangle with data $x = 5, y = 4, z = 10$ $x = 5, y = 4, z = 10$. The sum is $x + y < zx + y < z$ (not $x + y > zx + y > z$ as expected). Using these data from (1), we find the lengths of the sides $\Delta ABC \Delta ABC$ which need to construct: $a = y + z = 14a = y + z = 14$, $b = x + z = 15b = x + z = 15$, $c = x + y = 9c = x + y = 9$. And as can be seen the numbers $a = 14, b = 15, c = 9$ $a = 14, b = 15, c = 9$ can become side lengths $\Delta ABC \Delta ABC$.

Let us explain the reason for this phenomenon. From (8), for example, $x = (b + ca)/2$ $x = (b + ca)/2$. Here we see the difference between the sum of two sides and the third side. It is known that $b + c > ab + c > a$, but to what extent $b + c > ab + c > a$, there is no upper limit on this. This means that xx, as well as yy and zz, are independent of each other and can take on any positive value. In other words, unlike a, b, ca, b, c, the variables x, y, zx, y, z are free and can take on any positive value independently of each other. Therefore, we can expect that the variables x, y, z x, y, z have greater capabilities in solving problems than a, b, ca, b, c.

3) The quantities a, b, ca, b, c and x, y, zx, y, z connect two important figures of planimetry like a triangle and a circle (Figure 1). In the figure MNP are the tangent points, $OP = ON = OM = rOP = ON = OM = r$ are the radius of the inscribed circle, and $AM = AP = xAM = AP = x$, $BM = BN = yBM = BN = y$, $CN = CP = zCN = CP = z$. Then, from Figure 1 it is easy to obtain equalities (7).

Figure 1.

Visual example for Problem 3



Based on the above about the system of quantities a, b, ca, b, c and x, y, zx, y, z and Figure. 1 Lyapin et al., (1973) put forward the following idea: "if a problem on the topic triangles is presented in the language a, b, ca, b, c, it is first translated into the language x, y, zx, y, z using equalities (7)" and some necessary formulas with the hope that they received a more effective solution. Then, the authors provided a complete list of basic formulas in the language x, y, zx, y, z (with proof). Some formulas used in mixed language for convenience. We will use them here as needed related to a specific task. Let's return then to problems 1 and 2.

First, let us note one important point: Chernikov (2003) proved the equivalence of inequalities (1) and (2), but the validity of these inequalities remains open. Let us recall that in proving the equivalence of inequalities (1) and (2), Chernikov (2003) used equality (3), which raised some questions for us. Let's try to translate inequality (1) into the language x, y, zx, y, z using equalities (7). Inequality (1) then takes the form

$2x(y+z)^2+2y(x+z)^2+2z(x+y)^2 \leq 3(y+z)(x+z)(x+y)$

$2x(y+z)^2+2y(x+z)^2+2z(x+y)^2 \leq 3(y+z)(x+z)(x+y)$. By simple calculation and simplification, we can easily verify that inequality (1) leads to true inequalities of the form of (9)

$$6 \leq \frac{y}{z}+\frac{z}{y}+\frac{x}{z}+\frac{z}{x}+\frac{x}{y}+\frac{y}{z} \quad (9)$$

Considering that x, y, z are positive numbers, the validity of inequality (9) is beyond doubt. For inequality (2), first from Lyapin et al., (1973) we produce the following formulas $cos(A)=1-\frac{2yz}{bc} \quad cos(B)=1-\frac{2zx}{ac} \quad cos(C)=1-\frac{2xy}{ab}$.

$cos(A)=1-\frac{2yz}{bc} \quad cos(B)=1-\frac{2zx}{ac} \quad cos(C)=1-\frac{2xy}{ab}$. Then inequality (2) takes the following form:

$$1-\frac{2yz}{bc}+1-\frac{2zx}{ac}+1-\frac{2xy}{ab} \leq \frac{3}{2} \Rightarrow ayz+bzx+cxy \leq \frac{3}{4}abc \Rightarrow yz(y+z)+\ zx(x+z)+ xy(x+y) > \frac{3}{4}(y+z)(x+z)(x+y) \Rightarrow 6 \leq \frac{y}{z}+\frac{z}{y}+\frac{x}{z}+\frac{z}{x}+\frac{x}{y}+\frac{y}{z}$$

$$1-\frac{2yz}{bc}+1-\frac{2zx}{ac}+1-\frac{2xy}{ab} \leq \frac{3}{2} \Rightarrow ayz+bzx+cxy \leq \frac{3}{4}abc \Rightarrow yz(y+z)+\ zx(x+z)+ xy(x+y) > \frac{3}{4}(y+z)(x+z)(x+y) \Rightarrow 6 \leq \frac{y}{z}+\frac{z}{y}+\frac{x}{z}+\frac{z}{x}+\frac{x}{y}+\frac{y}{z}$$

And here this inequality is verified by ordinary calculation. The equivalence of inequalities (1) and (2) is obvious, since both of them, using the quantities x, y, zx, y, z, are reduced to the same inequality (9). Since (9) is a true inequality, we also prove the validity of inequalities (1) and (2) (this proof is not given by Chernikov (2003) in the language a, b, ca, b, c).

Identifying the equivalence of problems, especially when they have different structures, has great practical value. We often observe the equivalence of Olympiad and competitive problems in different books. Discovering the equivalence of a series of problems in time certainly has a positive effect on the content and structure of the books written. In one specific example, we saw that switching to language x, y, zx, y, z does not help us with this. Let's give one more example. İt is proposed to prove the inequality (10) where a, b, ca, b, c are the lengths of the sides of some $\Delta ABC\Delta ABC$.

$$(b+ca)(a+cb)(c+ab) < abc \quad (10)$$

Note that the circumscribed circle is equivalent. To prove these three inequalities, we need the following proven formulas from Lyapin et al., (1973):

$$sin\left(\frac{A}{2}\right) = \sqrt{\frac{yz}{bc}}\ , \quad sin\left(\frac{B}{2}\right) = \sqrt{\frac{xz}{ac}}\ , \quad sin\left(\frac{C}{2}\right) = \sqrt{\frac{xy}{ab}}\ (11)$$

In these formulas $\frac{a+b+c}{2}\frac{a+b+c}{2}$ is the semiperimeter of $\Delta ABC \Delta ABC$, then:

$$sin\left(\frac{A}{2}\right) \times sin\left(\frac{B}{2}\right) \times sin\left(\frac{c}{2}\right) = \sqrt{\frac{yz}{bc}} \times \sqrt{\frac{xz}{ac}} \times \sqrt{\frac{xy}{ab}} = \frac{xyz}{abc} \quad (12) \Rightarrow$$

$$\frac{xyz}{abc} \leq \frac{1}{8}\ (13) \Rightarrow abc \geq 8xyz \Rightarrow (x+y)(y+z)(x+z) \geq 8xyz\ (14)$$

It follows that

$$2rR \Longrightarrow 2\frac{S}{P} \leq \frac{abc}{4S} \Longrightarrow 8S^2 \leq abcp \Longrightarrow 8xyzp \leq abcp \Longrightarrow abc \geq 8xyz \Longrightarrow (14)\sim(12)$$

$2rR \Longrightarrow 2\frac{S}{P} \leq \frac{abc}{4S} \Longrightarrow 8S^2 \leq abcp \Longrightarrow 8xyzp \leq abcp \Longrightarrow abc \geq 8xyz \Longrightarrow (14)\sim(12)$. This means that all three inequalities (12-14) are equivalent to each other.

How many more equivalent problems there are in published works is a big question. In addition to proving the equivalence of statements, the variables x, y, zx, y, z can be effectively used in other problems, as well as in proving some theorems of planimetry. In Lyapin et al., (1973) about 20 main types of problems were selected from the topic of triangles.

An example is next, let the perimeter $\Delta ABC \Delta ABC$ be constant number, P=O. In what case does this triangle have a maximum area and find the value of this area. The solution is when the semi-perimeter: $p = \frac{a+b+c}{2} = x + y + z = \frac{c}{2}p = \frac{a+b+c}{2} = x + y + z = \frac{c}{2}$. According to the formula $S = \sqrt{xyzp}\ \ (16) S = \sqrt{xyzp}\ \ (16)$ where $x + y + z + p = \frac{c}{2}x + y + z + p = \frac{c}{2}$ is a constant number. Then, the product $xyzxyz$ has a maximum value when $x = y = z = \frac{c}{6} \Rightarrow S_{max} = \sqrt{\frac{c}{6} \times \frac{c}{6} \times \frac{c}{6} \times \frac{c}{2}} = \frac{c^2}{12\sqrt{3}} x = y = z = \frac{c}{6} \Rightarrow S_{max} = \sqrt{\frac{c}{6} \times \frac{c}{6} \times \frac{c}{6} \times \frac{c}{2}} = \frac{c^2}{12\sqrt{3}}$ and $\frac{c}{6} \Longrightarrow S_{max} = \sqrt{\frac{c}{6} \cdot \frac{c}{6} \cdot \frac{c}{6} \cdot \frac{c}{2}} = \frac{c^2}{12\sqrt{3}}$. And, if $x = y = zx = y = z$ then from (7) it follows that $a = b = c = \frac{c}{3} = b = c = \frac{c}{3}$, i.e., $\Delta ABC \Delta ABC$ is equilateral with side $a = \frac{c}{3} a = \frac{c}{3}$ and $S_{max} = \frac{c^2}{12\sqrt{3}}$ $S_{max} = \frac{c^2}{12\sqrt{3}}$.

It often happens that the proposed problem on the topic of a triangle is difficult to solve in the language a, b, ca, b, c. Either it is difficult to choose a solution method, or it is necessary to prove some preliminary statements. These difficulties are probably because the quantities a, b, ca, b, c satisfy the triangle inequalities. In such cases, they often switch to language x, y, zx, y, z and this transition basically justifies itself, because they are free variables (positive). Let's analyze another example.

<u>Problem 4.</u> In triangle ABC, prove the inequality (17) if $a + b + c = 2a + b + c = 2$, where a, b, c a, b, c are the lengths of the sides $\Delta ABC\Delta ABC$.

$a^2 + b^2 + c^2 < 2(1 - abc) \quad (17)$

In the solution, the author proposes to first prove the following preliminary equalities (18) from (1) where rr and RR are the radii of the inscribed and circumscribed circle, respectively.

$$abc = 4prRabc = 4prR; \; ab + bc + ac = r2 + p2 + 4rR \quad (18)$$

$$ab + bc + ac = r2 + p2 + 4rR \quad (18)$$

The further part of this solution is naturally connected with these unexpected equalities. This constitutes additional difficulties, and these difficulties are also mysterious, since the need for equalities (18) in the solution is unfounded from the very beginning of the solution. Other similar questions rise here but one thing is clear - this solution cannot be considered the most effective solution. What happens if we fail to guess the equalities in (18)?

If we move to x, y, zx, y, z, we will immediately notice that from the additional condition $a + b + c = 2a + b + c = 2$ it follows that $p = x + y + z = 1p = x + y + z = 1$. Hence, $a = y + z = 1 - xa = y + z = 1 - x$, $b = x + z = 1 - yb = x + z = 1 - y$, $c = x + y = 1 - z$ $c = x + y = 1 - z$ and inequality (17) will take the form $(1 - x)^2 + (1 - y)^2 + (1 - z)^2 < 2[1 - (1 - x)(1 - y)(1 - z)]$ $(1 - x)^2 + (1 - y)^2 + (1 - z)^2 < 2[1 - (1 - x)(1 - y)(1 - z)]$. When simplifying this inequality, we will take into account that $p = 1p = 1$ as shown next:

$$1 - 2x + x^2 + 1 - 2y + y^2 + 1 - 2z + z^2 < 2[1 - (1 - y - x + xy - z + zy + zx - xyz)]$$
$$\Rightarrow 3 - 2(x + y + z) + x^2 + y^2 + z^2 < 2(x + y + z - xy - zy - zx + xyz)$$
$$\Rightarrow 1 + (x^2 + y^2 + z^2 + 2xy + 2zy + 2zx) < 2 + 2xyz$$
$$\Rightarrow 1 + (x + y + z)^2 < 2 + 2xyz \Rightarrow 2 < 2xyz \Rightarrow xyz > 0$$

We have obtained a correct inequality, which means that inequality (17) with the condition $a + b + c = 2a + b + c = 2$ is also true. Here it is considered that during the calculation we made equivalent transformations. We wonder where equalities (18) that were necessary to solve this problem disappeared? Let's look at another example.

Problem 5 (XLV Moscow Olympiad, 1982). Prove that

$$a^4+b^4+c^4-2(a^2b^2+a^2c^2+b^2c^2)\ +a^2bc+b^2ac+c^2ab\geq 0$$

$$a^4+b^4+c^4-2(a^2b^2+a^2c^2+b^2c^2)\ +a^2bc+b^2ac+c^2ab\geq 0$$ (19).

If we denote the left side of this inequality by QQ, then by direct calculation we will arrive at (20).

$$Q=(a+b+c)\,[abc-(b+ca)(a+cb)(b+ac)]\ \ (20)$$

After this, based on inequality (10), inequality (19) is obvious." But again the same question: how to find (20)? What will we do if we fail to guess this identity? Moving to x,y,zx,y,z (19) is proved without (20) [see problems 5-9 in Lyapin et al., (1973)].

On the other hand, the nature of the variables x, y, z sometimes makes it possible to apply the methods of higher mathematics to solving some problems from school mathematics as in the next example.

Problem 6 (XXIV Olympiad, Moldova, 1980). When in triangle ABC the condition $abc=1\ (21)abc=1\ (21)$ is satisfied. Find the maximum area of this triangle if a,b,ca,b,c are the lengths of the sides of this triangle.

If we switch to the language, x,y,zx,y,z we must find the maximum of the function $S=\sqrt{xyzp}=\sqrt{xyz(x+y+z)}S=\sqrt{xyzp}=\sqrt{xyz(x+y+z)}$ considering $(x>0,y>0,z>0)(x>0,y>0,z>0)$ provided that $(x+y)(x+z)(y+z)=1(x+y)(x+z)(y+z)=1$. Further, the solution is completed using the conditional extremum method [see task 18.9 in Lyapin et al., (1973)].

In addition, sometimes you have to make the reverse transition, i.e., if the problem given in the language x,y,zx,y,z these are arbitrary positive quantities. If we use formulas in (8) we switch to the language a,b,ca,b,c and obtain the corresponding equivalent problem on the topic of triangles. The resulting statement does not require additional proof. An example is shown in the next problem.

Problem 7. Proof that for any positive variables x,y,zx,y,z the following inequality is true

$$(xy)^2(x+yz)+(yz)^2(y+zx)+(zx)^2(z+xy)\ \geq 0\ \ (22)$$

$$(xy)^2(x+yz)+(yz)^2(y+zx)+(zx)^2(z+xy)\ \geq 0\ \ (22)$$.

By finding the corresponding equivalent problem in language a,b,ca,b,c on the topic of triangles we arrive at (23).

$$(ba)^2(3c-ab)+(cb)^2(3a-bc)+(ac)^2(3b-ac)\geq 0\ (23)$$

Inequality (23) is complex, and it will probably not be easy to prove this inequality without (22). We encourage the reader to take it on as a challenge.

Conclusions

The proof of equivalent problems in the geometry of triangles plays an essential role in the field of mathematics and has profound relevance both in mathematical theory and in its practical applicability. In our opinion, the proof of equivalent problems in the geometry of triangles is essential to establish a solid foundation in mathematics since the methods of proof encourage critical thinking and the ability to reason logically. Those involved in proving equivalent problems develop analytical skills that are valuable not only in mathematics, but also in a wide range of academic and professional disciplines. In terms of practical applicability these demonstrations have direct relevance in engineering, physics, architecture, cartography, and many other areas. Throughout this work, various aspects of this area have been explored, highlighting the challenges involved in its analysis. Considering everything that was discussed, we can say that the question we raise is serious in nature. How many other equivalent problems exist in published works? How to find them all?

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Conflicto de intereses

El autor declara que no existe conflicto de intereses